Model Hierarchy

→ The Vlasov Eqn. (1D)

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0$$

$$\underline{E} = -\partial_x \phi, \quad \nabla^2\phi = -4\pi\int dv\, f$$

is collisionless Boltzmann Eqn.

$$\underbrace{\frac{\partial f}{\partial t}}_{\omega} + \underbrace{\underline{v}\cdot\frac{\partial f}{\partial \underline{x}}}_{v_{Th}/L \sim k v_{Th}} + \underbrace{\frac{q}{m}\underline{E}\cdot\frac{\partial f}{\partial \underline{v}}}_{\frac{q}{m}E/v_{Th}} = \underbrace{C(f,f)}_{\nu}$$

i.e. applicable to problems where:

$$\omega, \; v_{Th}/L \gg \nu.$$

In practise, this means:

- take $f = \underbrace{\langle f_{equ}\rangle}_{\text{unperturbed}} + \underbrace{\delta f}_{\text{perturbation}}$
- $\langle f_{eq}\rangle$ set by collisional processes

<u>i.e.</u> l.o. $C(f) \cong 0 \Rightarrow \langle f_0\rangle$ is local Maxwellian.

$f(x,v,t) \equiv$ all info. Any macroscopic quantity via moments.

- In case where perturbations fast relative to collision time, i.e.

$\omega,\ k v_{Th} > \nu$, Vlasov Equation

applies to perturbation dynamics.

- Vlasov Equation is statement of conservation of phase space density along particle orbits

i.e. $f \Leftrightarrow$ phase space density

$$\frac{\partial f}{\partial t} + \underbrace{v}_{dx/dt} \frac{\partial f}{\partial x} + \underbrace{\frac{q}{m} E}_{dv/dt} \frac{\partial f}{\partial v} = 0$$

characteristic Eqns

then

$$\frac{\partial f}{\partial t} + \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dv}{dt}\frac{\partial f}{\partial v} = 0 = \frac{df}{dt}$$

obviously collisions violate phase space density conservation.

- obvious analogy:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \underline{\nabla} \rho = - \rho \, \underline{\nabla} \cdot \underline{v}$$

For $\underline{\nabla} \cdot \underline{v} = 0$, incompressible flow;

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \underline{\nabla} \rho = \frac{d\rho}{dt} = 0$$

$\rho(x, t)$ const. along trajectory $\frac{dx}{dt} = \underline{v}(x, t)$

- Connection:

For Hamiltonian system, Liouville's Thm (phase volume conservation) implies

$\underline{\nabla}_\Gamma \cdot \underline{V}_\Gamma = 0 \rightarrow$ flow in phase space is incompressible

$$\underline{V}_\Gamma = \left(\frac{dx}{dt}, \frac{dp}{dt} \right)$$

$$\underline{\nabla} \cdot \underline{V}_\Gamma = \frac{\partial}{\partial x}\frac{dx}{dt} + \frac{\partial}{\partial p}\frac{dp}{dt} = \cancel{\frac{\partial V}{\partial x}} + \cancel{\frac{\partial}{\partial p}} qE = 0.$$

→ Where does it come from?

- Hamilton's Eqns → N particles

$$\frac{dx_i}{dt} = v_i = p_i/m$$

$$\frac{dp_i}{dt} = -\nabla_i \left(\sum_{j \neq i} q/|x_i - x_j| \right)$$

⇒ Liouville's Eqn:

$f_N = f_N(t, x_1, v_1, \dots x_N, v_N)$ → N-body distribution

$$\frac{\partial f_N}{\partial t} + \{H, f_N\} = 0$$

How get from N-body distribution equation to 1, or 2 body equation?

- BBGKY hierarchy - successive integration out

$$\frac{\partial f_{N-1}}{\partial t} + L_{N-1} f_{N-1} = -\int d\Gamma_N L_N f_N$$

⋮

$$\frac{\partial f_2}{\partial t} + L_2 f_2 = -\int d\Gamma_3 L_3 f_3$$

and:

$$\underbrace{\frac{\partial f_1}{\partial t} + L_1 f_1}_{\text{Vlasov Eqn.}} = \underbrace{-\int d\Gamma_2\, L_2 f_2}_{\substack{C(f_1, f_2) \\ \text{collision operator}}}$$

note:

$$C(f) = \int dV_2 \, L_2 \, f(1,2)$$

$$= \int dV_2 \, L_2 \, f(1)\, f(2)$$

collision operator for a test particle

What allows truncation, factorization?

$\Rightarrow$ $\boxed{\text{Diluteness}}$ $\rightarrow$ $\left\{ \begin{array}{l} T > e^2/\bar{r} \\ \text{or} \\ 1/n\lambda_D^3 \ll 1 \end{array} \right.$

- allows neglect $\int f_3$
- allows factorization $f(1,2) \rightarrow f(1)\, f(2)$

$\Rightarrow$ weak correlation expansion.

→ For neutral gas, same story with expansion parameter:

$$h^3/\bar{r}^3 = nh^3 < 1$$

ratio $\left(\text{range}/\text{mean inter-particle distance}\right)^3$.

→ In practice, dilute plasma phenomena described by:

① → Boltzmann / Fokker-Planck Eqn. (Landau)

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f + \frac{q}{m} \underline{E} \cdot \frac{\partial f}{\partial \underline{v}} = C(f)$$

+ Maxwell eqns. with sources linked to f.

for Coulomb force, approach via Landau (F-P E) operator

② → Vlasov Eqn.

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{q}{m} E \frac{\partial f}{\partial v} = 0 \qquad \text{(1D)}$$

⇒ <u>N.B.</u>

— For stellar-dynamics, Vlasov equation applies to both "equilibrium" and perturbations

<u>i.e.</u> system collisionless on all scales.

— "Equilibrium" via BGK solutions, Violent Relaxation, <u>etc.</u>

c.f. Binney and Tremaine.